Physics 606 Final Exam Solution

1. This problem is done in Baym, pp. 228–229, and was done in class. [The other problems are original.]

2. Use (12-12) of Baym:

$$P_{0\to 1} = \left| \frac{1}{i\hbar} \int_{-\infty}^{\infty} dt' \, e^{i(\hbar\omega)t'/\hbar} \langle 1 | V_{t'} | 0 \rangle \right|^2$$

with $V_t = q \frac{\mathcal{E}}{\sqrt{\pi}\,\tau} e^{-(t/\tau)^2} \sqrt{\frac{\hbar}{2m\omega}} \, (a + a^+)$

and $\langle 1 | (a + a^+) | 0 \rangle = 0 + \langle 1 | 1 \rangle = 1$

Then $P_{0\to 1} = \frac{1}{\hbar^2} q^2 \frac{\mathcal{E}^2}{\pi\tau^2} \frac{\hbar}{2m\omega} \left| \underbrace{\int_{-\infty}^{\infty} dt \, e^{i\omega t} e^{-t^2/\tau^2}}_{= I} \right|^2$

From given Gaussian integral, with $a = \frac{1}{\tau^2}$, $b = i\omega$, $c = 0$,

$$I = \sqrt{\frac{\pi}{1/\tau^2}} \, e^{-\frac{\omega^2}{4/\tau^2}} = \sqrt{\pi}\,\tau \, e^{-\frac{\omega^2\tau^2}{4}}$$

so $\boxed{P_{0\to 1} =} \frac{1}{\hbar^2} q^2 \frac{\mathcal{E}^2}{\pi\tau^2} \frac{\hbar}{2m\omega} \cdot \pi\tau^2 e^{-\frac{\omega^2\tau^2}{2}}$

$$= \boxed{\frac{q^2 \mathcal{E}^2}{2m\hbar\omega} e^{-\frac{\omega^2\tau^2}{2}}}$$

3. (a) $\hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1$ ($\hat{N}$ is Hermitian)

$\Rightarrow (e^{i\hat{\phi}}\sqrt{\hat{N}}\, e^{-i\omega t})(e^{+i\omega t}\sqrt{\hat{N}}\, e^{-i\hat{\phi}}) - (e^{+i\omega t}\sqrt{\hat{N}}\, e^{-i\hat{\phi}})(e^{i\hat{\phi}}\sqrt{\hat{N}}\, e^{-i\omega t}) = 1$

$\Rightarrow \boxed{e^{i\hat{\phi}}\hat{N} - \hat{N}e^{i\hat{\phi}} = e^{i\hat{\phi}}}$

(b) Since $e^{i\hat{\phi}} = \sum_{n=0}^{\infty} \frac{(i\hat{\phi})^n}{n!}$, the above equation is

$$\sum_{n=1}^{\infty} \frac{i^n}{n!} (\hat{\phi}^n \hat{N} - \hat{N}\hat{\phi}^n) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \hat{\phi}^n \quad \text{[with } (n = 0 \text{ term on left}) = \hat{N} - \hat{N} = 0 \text{]}$$

$$= \sum_{n=1}^{\infty} \frac{i^{n-1}}{(n-1)!} \hat{\phi}^{n-1}$$

which will be satisfied if

$\frac{i^n}{n!} (\hat{\phi}^n \hat{N} - \hat{N}\hat{\phi}^n) = \frac{i^{n-1}}{(n-1)!} \hat{\phi}^{n-1}$ or $\boxed{\hat{N}\hat{\phi}^n - \hat{\phi}^n\hat{N} = n i \hat{\phi}^{n-1}}$.

$[\hat{N}, \hat{\phi}] = i$ is assumed

$\underline{n=1}$: $\hat{N}\hat{\phi} - \hat{\phi}\hat{N} = i$ or

proof by induction: If the above equation is true for n,

$\hat{N}\hat{\phi}^{n+1} = \hat{\phi}^n\hat{N}\hat{\phi} + ni\hat{\phi}^{n-1}\hat{\phi} = \hat{\phi}^n(\hat{\phi}\hat{N} + i) + ni\hat{\phi}^n = \hat{\phi}^{n+1}\hat{N} + (n+1)i\hat{\phi}^n$

or $\hat{N}\hat{\phi}^{n+1} - \hat{\phi}^{n+1}\hat{N} = (n+1)\,i\,\hat{\phi}^n$,

so true for all n, QED

(c) Our proof of $\Delta x \Delta p_x \geq \frac{1}{2}\hbar$ followed from $[\hat{x}, \hat{p}_x] = i\hbar$,
so it still holds here, with $\boxed{\Delta N \Delta\phi \geq \frac{1}{2}}$.

4. (a) As in (9-51) and (9-23) of Baym,

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \quad , \quad f(\theta) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, V(r)$$

Here $f(\theta) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int_0^\infty dr\, r^2 \underbrace{\int_0^{2\pi} d\phi}_{=2\pi} \int_0^\pi d\theta \sin\theta\, V_0 e^{-r^2/a^2} e^{iqr\cos\theta}$

$$= -\frac{m}{\hbar^2} V_0 \int_0^\infty dr\, r^2 e^{-r^2/a^2} \underbrace{\int_1^{-1} (-d\mu)\, e^{iqr\mu}}_{=\left[\frac{e^{iqr\mu}}{iqr}\right]_{-1}^{1} = \frac{2}{qr}\sin(qr)} \qquad \mu = \cos\theta$$

$$= -\frac{2m}{\hbar^2}\frac{V_0}{q}\underbrace{\int_0^\infty dr\, r\, e^{-r^2/a^2}\sin qr}_{=\frac{1}{4}\sqrt{\pi}\, q a^3 e^{-a^2q^2/4} \text{ from given integral}}$$

$$= -\frac{2m}{\hbar^2} V_0 \frac{\sqrt{\pi}\, a^3}{4} e^{-a^2q^2/4}$$

so $$\boxed{\frac{d\sigma}{d\Omega} = \frac{\pi}{4}\frac{m^2}{\hbar^4} a^6 V_0^2 e^{-a^2q^2/2}}$$

(b) As in (9-34) of Baym,

$$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\theta)\, e^{i\delta_\ell}\sin\delta_\ell$$

Now use $\int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta = \frac{2}{2\ell+1}\delta_{\ell\ell'}$,

choosing $\ell = 0$, with $P_0 = 1$:

$$\frac{2}{k} e^{i\delta_0}\sin\delta_0 = \int_0^\pi f(\theta)\sin\theta\, d\theta = -\frac{2m}{\hbar^2}V_0\frac{\sqrt{\pi}\,a^3}{4}\underbrace{\int_0^\pi e^{-a^2(4k^2\sin^2\frac{\theta}{2})/4}\sin\theta\, d\theta}_{=I}$$

Let $\alpha = \frac{\theta}{2}$: $\sin(2\alpha) = 2\sin\alpha\cos\alpha = \frac{d}{d\alpha}(\sin^2\alpha)$, so

$$I = \int_0^{\pi/2} e^{-a^2k^2\sin^2\alpha}\frac{d}{d\alpha}(\sin^2\alpha)\, d(2\alpha) = 2\underbrace{\int_0^1 e^{-a^2k^2u}\, du}_{=\left[\frac{e^{-a^2k^2u}}{-a^2k^2}\right]_0^1}$$

$$= \frac{2}{a^2k^2}\left(1 - e^{-a^2k^2}\right)$$

and

$$\frac{2}{k}e^{i\delta_0}\sin\delta_0 = -\frac{2m}{\hbar^2}V_0\frac{\sqrt{\pi}\,a^3}{4}\cdot\frac{2}{a^2k^2}\left(1-e^{-a^2k^2}\right) = -\frac{mV_0 a\sqrt{\pi}}{\hbar^2k^2}\left(1-e^{-a^2k^2}\right)$$

$$\rightarrow -\frac{mV_0 a\sqrt{\pi}}{\hbar^2k^2}\left(1-(1-a^2k^2)\right) = -2A \quad , \quad \boxed{A \equiv \frac{mV_0a^3\sqrt{\pi}}{2\hbar^2}}$$

As $k\to 0$, $e^{i\delta_0}\sin\delta_0 \to -Ak$ so $e^{i\delta_0}\sin\delta_0 \approx \delta_0$, and $\boxed{\delta_0 \to -Ak}$

Check: $\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0$ as in (9-37) and $\sigma = 4\pi\frac{d\sigma}{d\Omega} \approx \sigma_0$ as $k\to 0$,

so $\delta_0^2 \approx k^2\cdot\frac{\pi}{4}\frac{m^2}{\hbar^4}a^6V_0^2\cdot 1 = A^2k^2 \Rightarrow \delta_0 = \pm Ak$ & repulsive $\Rightarrow -$,

so $\boxed{\delta_0 \to -Ak}$.